P Preparation for Calculus





P.4 Review of Trigonometric Functions

Objectives

- Describe angles and use degree measure.
- Use radian measure.
- Understand the definitions of the six trigonometric functions.
- Evaluate trigonometric functions.
- Solve trigonometric equations.
- Graph trigonometric functions.

Angles and Degree Measure

Angles and Degree Measure

An **angle** has three parts: an **initial ray** (or side), a **terminal ray,** and a **vertex** (the point of intersection of the two rays), as shown in Figure P.32(a).

An angle is in **standard position** when its initial ray coincides with the positive *x*-axis and its vertex is at the origin, as shown in Figure P.32(b).

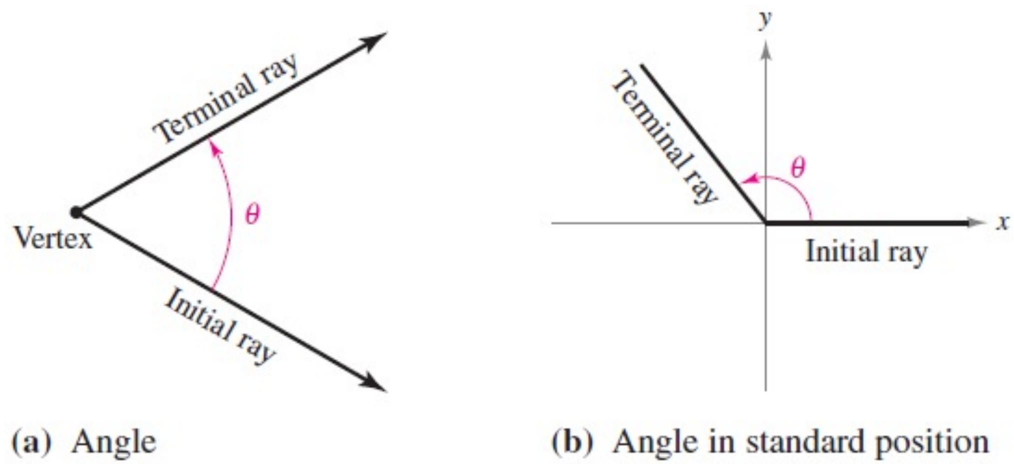


Figure P.32

Angles and Degree Measure

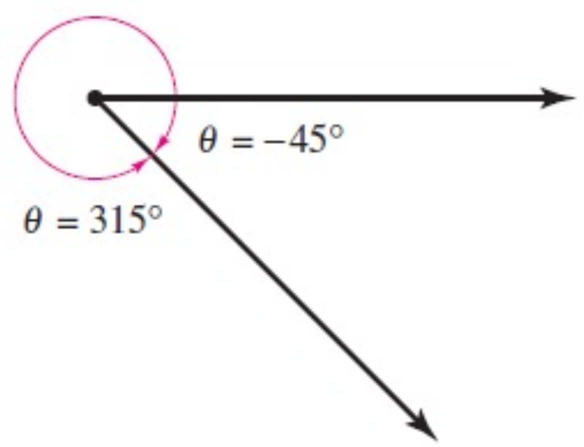
It is assumed that you are familiar with the degree measure of an angle.

It is common practice to use θ (the lowercase Greek letter theta) to represent both an angle and its measure.

Angles between 0° and 90° are **acute,** and angles between 90° and 180° are **obtuse.**

Angles and Degree Measure

Positive angles are measured *counterclockwise*, and negative angles are measured *clockwise*. For instance, Figure P.33 shows an angle whose measure is −45°.



Coterminal angles

Figure P.33

Angles and Degree Measure

You cannot assign a measure to an angle by simply knowing where its initial and terminal rays are located. To measure an angle, you must also know how the terminal ray was revolved.

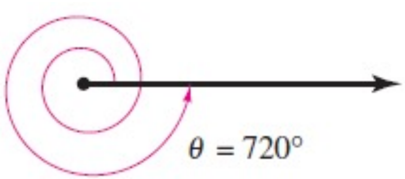
For example, Figure P.33 shows that the angle measuring $-45°$ has the same terminal ray as the angle measuring $315°$. Such angles are **coterminal.**

In general, if θ is any angle, then $\theta + n(360)$, n is a nonzero integer, is coterminal with θ.

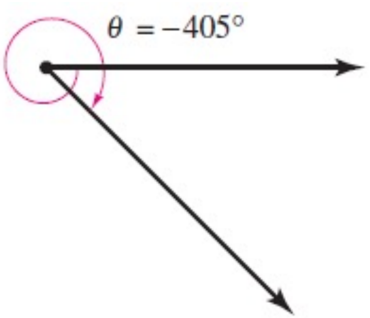
Angles and Degree Measure

An angle that is larger than 360° is one whose terminal ray has been revolved more than one full revolution counterclockwise, as shown in Figure P.34(a).

You can form an angle whose measure is less than −360° by revolving a terminal ray more than one full revolution clockwise, as shown in Figure P.34(b).



(a) An angle whose measure is greater than 360°



(b) An angle whose measure is less than −360°

Figure P.34

Radian Measure

Radian Measure

To assign a radian measure to an angle θ, consider θ to be a central angle of a circle of radius 1, as shown in Figure P.35.

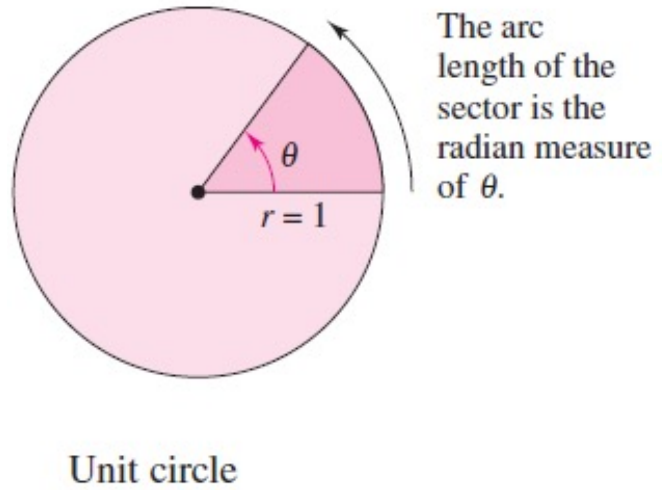


Figure P.35

The **radian measure** of θ is then defined to be the length of the arc of the sector. Because the circumference of a circle is $2\pi r$, the circumference of a **unit circle** (of radius 1) is 2π.

Radian Measure

This implies that the radian measure of an angle measuring 360° is 2π. In other words, 360° = 2π radians.

Using radian measure for θ, the length s of a circular arc of radius r is $s = r\theta$, as shown in Figure P.36.

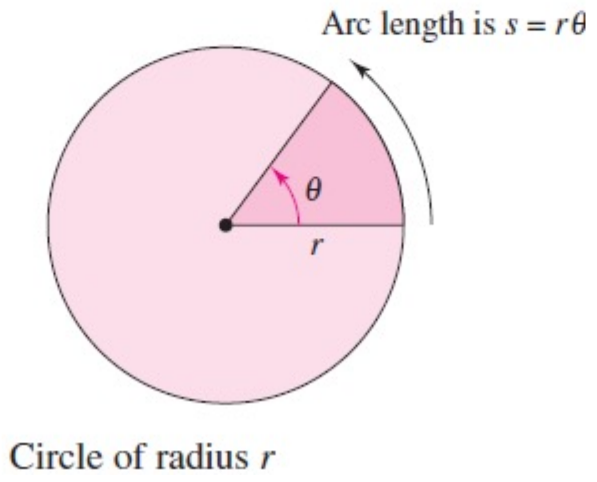
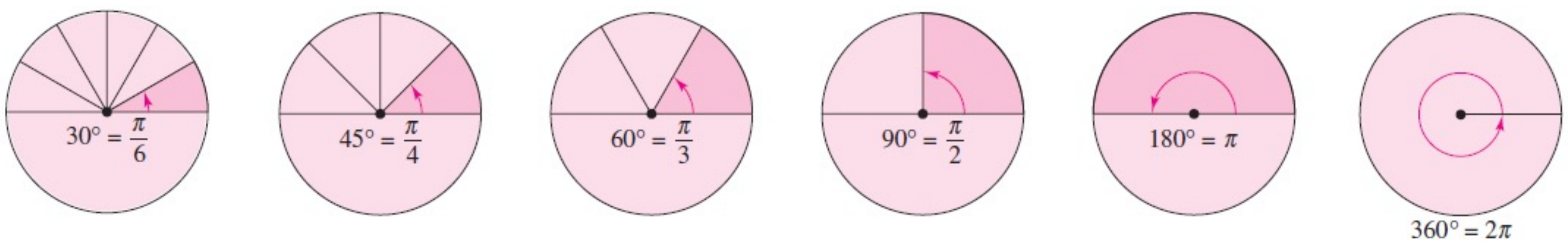


Figure P.36

Radian Measure

You should know the conversions of the common angles shown in Figure P.37.

For other angles, use the fact that 180° is equal to π radians.



Radian and degree measures for several common angles

Figure P.37

Example 1 – *Conversions Between Degrees and Radians*

a. $40^\circ = (40\ \cancel{\text{deg}})\left(\dfrac{\pi \text{ rad}}{180\ \cancel{\text{deg}}}\right) = \dfrac{2\pi}{9}\text{ radian}$

b. $540^\circ = (540\ \cancel{\text{deg}})\left(\dfrac{\pi \text{ rad}}{180\ \cancel{\text{deg}}}\right) = 3\pi \text{ radians}$

c. $-270^\circ = (-270\ \cancel{\text{deg}})\left(\dfrac{\pi \text{ rad}}{180\ \cancel{\text{deg}}}\right) = -\dfrac{3\pi}{2}\text{ radians}$

d. $-\dfrac{\pi}{2}\text{ radians} = \left(-\dfrac{\pi}{2}\ \cancel{\text{rad}}\right)\left(\dfrac{180 \text{ deg}}{\pi\ \cancel{\text{rad}}}\right) = -90^\circ$

Example 1 – *Conversions Between Degrees and Radians*

cont'd

e. $$2 \text{ radians} = (2 \cancel{\text{rad}})\left(\frac{180 \text{ deg}}{\pi \cancel{\text{rad}}}\right)$$

$$= \left(\frac{360}{\pi}\right)^{\circ} \approx 114.59^{\circ}$$

f. $$\frac{9\pi}{2} \text{ radians} = \left(\frac{9\pi}{2} \cancel{\text{rad}}\right)\left(\frac{180 \text{ deg}}{\pi \cancel{\text{rad}}}\right)$$

$$= 810^{\circ}$$

The Trigonometric Functions

The Trigonometric Functions

There are two common approaches to the study of trigonometry. In one, the trigonometric functions are defined as ratios of two sides of a right triangle.

In the other, these functions are defined in terms of a point on the terminal ray of an angle in standard position.

The six trigonometric functions, **sine, cosine, tangent, cotangent, secant,** and **cosecant** (abbreviated as sin, cos, tan, cot, sec, and csc, respectively), are defined from both viewpoints.

The Trigonometric Functions

Definition of the Six Trigonometric Functions

Right triangle definitions, where $0 < \theta < \frac{\pi}{2}$ (see Figure P.38)

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

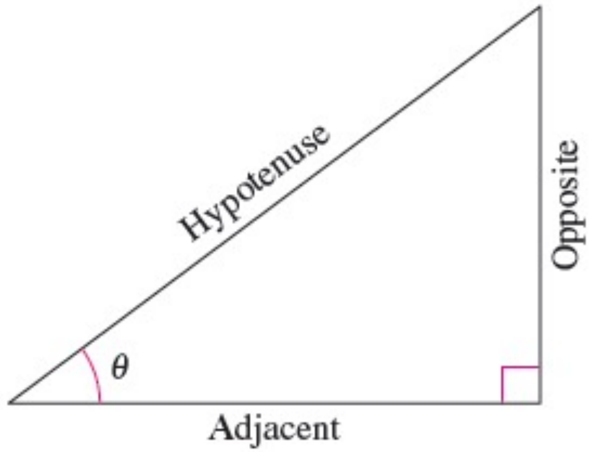
$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

Circular function definitions, where θ *is any angle* (see Figure P.39)

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x},\ x \neq 0$$

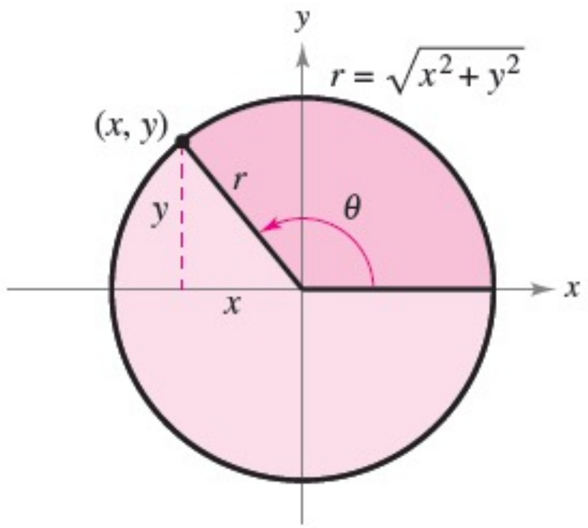
$$\csc\theta = \frac{r}{y},\ y \neq 0 \qquad \sec\theta = \frac{r}{x},\ x \neq 0 \qquad \cot\theta = \frac{x}{y},\ y \neq 0$$

The Trigonometric Functions



Sides of a right triangle

Figure P.38



An angle in standard position

Figure P.39

The Trigonometric Functions

The trigonometric identities listed below are direct consequences of the definitions.

TRIGONOMETRIC IDENTITIES

Pythagorean Identities

$\sin^2\theta + \cos^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

Even/Odd Identities

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

$\csc(-\theta) = -\csc\theta$

$\sec(-\theta) = \sec\theta$

$\cot(-\theta) = -\cot\theta$

Sum and Difference Formulas

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$$

$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$

$$\tan(\theta \pm \phi) = \frac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta\tan\phi}$$

Power-Reducing Formulas

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

$$\tan^2\theta = \frac{1 - \cos 2\theta}{1 + \cos 2\theta}$$

Double-Angle Formulas

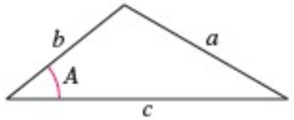
$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 1 - 2\sin^2\theta\\ &= \cos^2\theta - \sin^2\theta\end{aligned}$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Law of Cosines

$$a^2 = b^2 + c^2 - 2bc\cos A$$



Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}$$

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

Evaluating Trigonometric Functions

Evaluating Trigonometric Functions

There are two ways to evaluate trigonometric functions:

(1) decimal approximations with a graphing utility and

(2) exact evaluations using trigonometric identities and formulas from geometry.

When using a graphing utility to evaluate a trigonometric function, remember to set the graphing utility to the appropriate mode—*degree* mode or *radian* mode.

Example 2 – *Exact Evaluation of Trigonometric Functions*

Evaluate the sine, cosine, and tangent of $\pi/3$.

Solution:

Because 60° = $\pi/3$ radians, you can draw an equilateral triangle with sides of length 1 and θ as one of its angles, as shown in Figure P.40.

Because the altitude of this triangle bisects its base, you know that $x = \frac{1}{2}$.

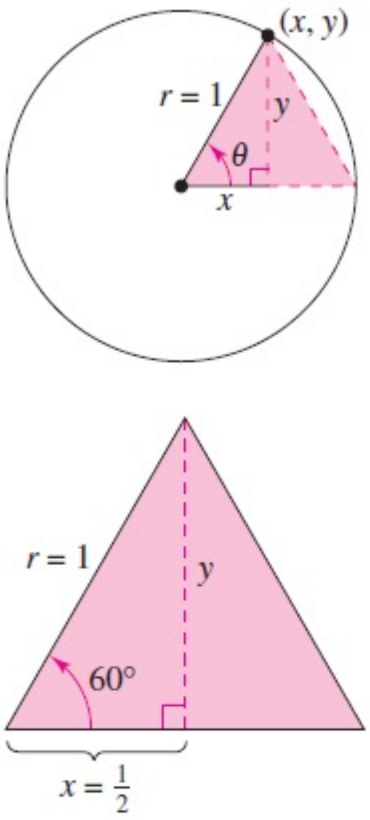


Figure P.40

Example 2 – *Solution*

cont'd

Using the Pythagorean Theorem, you obtain,

$$y = \sqrt{r^2 - x^2} = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}.$$

Now, knowing the values of *x*, *y*, and *r*, you can write the following.

$$\sin\frac{\pi}{3} = \frac{y}{r} = \frac{\sqrt{3}/2}{1} = \frac{\sqrt{3}}{2}$$

$$\cos\frac{\pi}{3} = \frac{x}{r} = \frac{1/2}{1} = \frac{1}{2}$$

$$\tan\frac{\pi}{3} = \frac{y}{x} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

Evaluating Trigonometric Functions

The degree and radian measures of several common angles are shown in the table below, along with the corresponding values of the sine, cosine, and tangent

Trigonometric Values of Common Angles

θ (degrees)	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$
θ (radians)	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined	0	Undefined

Evaluating Trigonometric Functions

See Figure P.41.

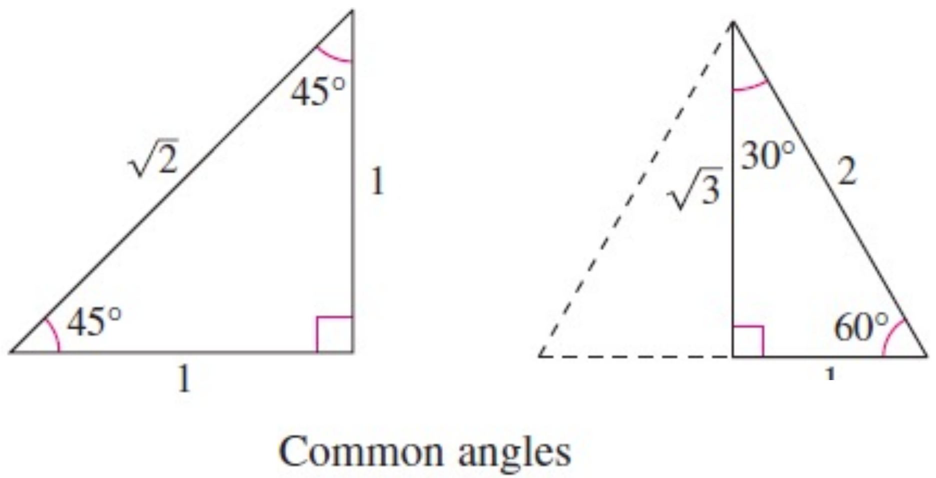


Figure P.41

Example 3 – *Using Trigonometric Identities*

a. $\sin\left(-\frac{\pi}{3}\right) = -\sin\frac{\pi}{3}$ $\quad\quad$ $\sin(-\theta) = -\sin\theta$

$= -\frac{\sqrt{3}}{2}$

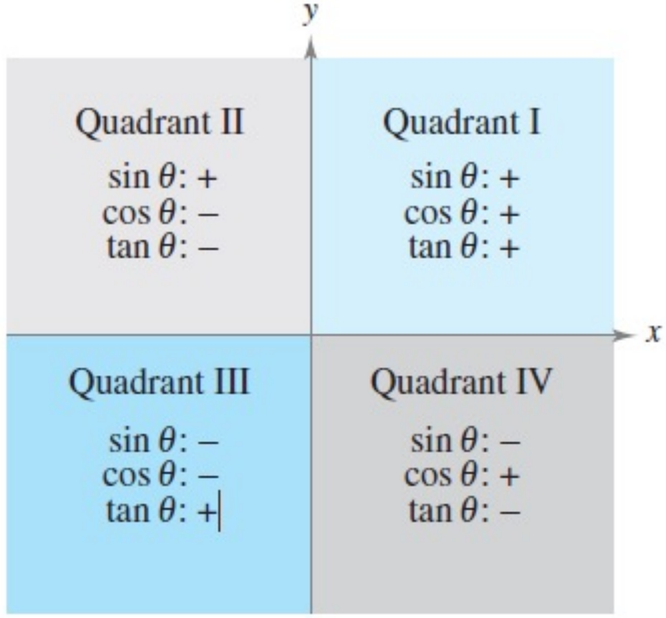
b. $\sec 60° = \frac{1}{\cos 60°}$ $\quad\quad$ $\sec\theta = \frac{1}{\cos\theta}$

$= \frac{1}{1/2}$

$= 2$

Evaluating Trigonometric Functions

The quadrant signs for the sine, cosine, and tangent functions are shown in Figure P.42.

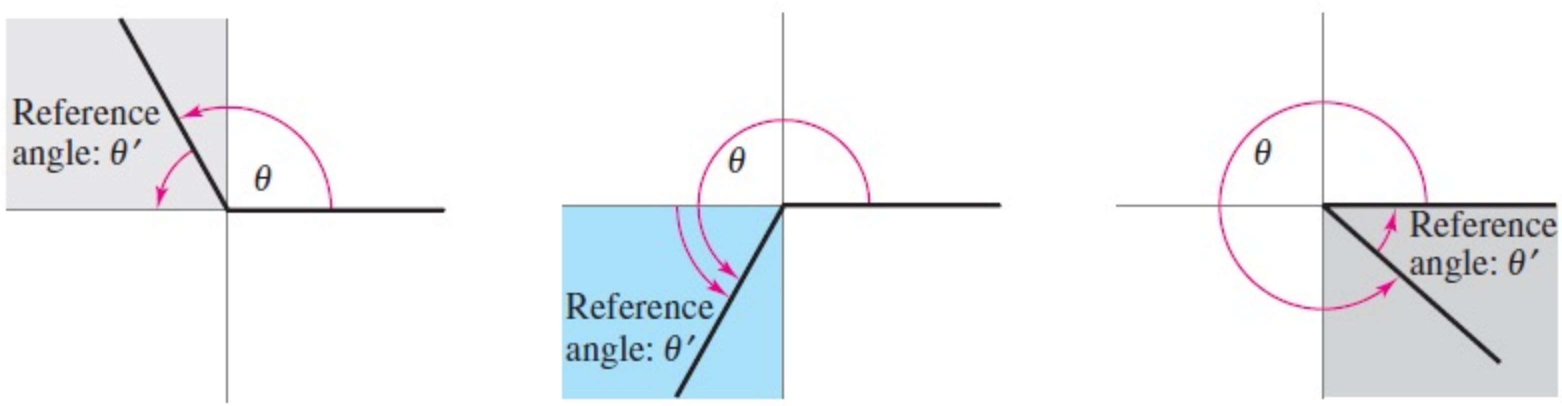


Quadrant signs for trigonometric functions

Figure P.42

Evaluating Trigonometric Functions

To find the angles in quadrants other than the first quadrant, you can use the concept of a **reference angle** (see Figure P.43), with the appropriate quadrant sign.



Quadrant II
$\theta' = \pi - \theta$ (radians)
$\theta' = 180° - \theta$ (degrees)

Quadrant III
$\theta' = \theta - \pi$ (radians)
$\theta' = \theta - 180°$ (degrees)

Quadrant IV
$\theta' = 2\pi - \theta$ (radians)
$\theta' = 360° - \theta$ (degrees)

Figure P.43

Evaluating Trigonometric Functions

For instance, the reference angle for $3\pi/4$ is $\pi/4$, and because the sine is positive in Quadrant II, you can write

$$\sin \frac{3\pi}{4} = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}.$$

Similarly, because the reference angle for 330° is 30°, and the tangent is negative in Quadrant IV, you can write

$$\tan 330^\circ = -\tan 30^\circ = -\frac{\sqrt{3}}{3}.$$

Solving Trigonometric Equations

Solving Trigonometric Equations

How would you solve the equation $\sin\theta = 0$?

You know that $\theta = 0$ is one solution, but this is not the only solution. Any one of the following values of θ is also a solution.

$$\ldots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots$$

You can write this infinite solution set as $\{n\pi : n \text{ is an integer}\}$.

Example 4 – *Solving a Trigonometric Equation*

Solve the equation $\sin\theta = -\frac{\sqrt{3}}{2}$.

Solution:

To solve the equation, you should consider that the sine function is negative in Quadrants III and IV and that

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

So, you are seeking values of θ in the third and fourth quadrants that have a reference angle of $\pi/3$.

Example 4 – *Solution*

cont'd

In the interval $[0, 2\pi]$, the two angles fitting these criteria are,

$$\theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3} \quad \text{and} \quad \theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}.$$

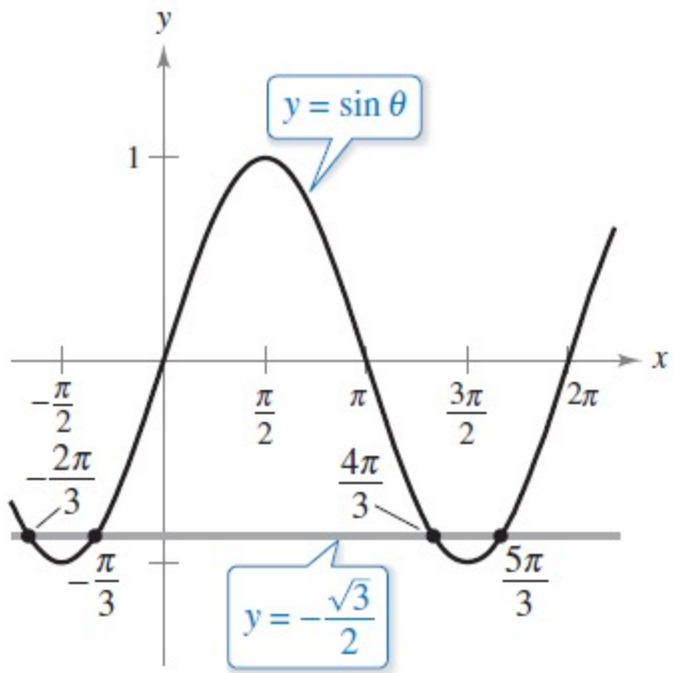
By adding integer multiples of 2π to each of these solutions, you obtain the following general solution.

$$\theta = \frac{4\pi}{3} + 2n\pi \quad \text{or} \quad \theta = \frac{5\pi}{3} + 2n\pi, \quad \text{where } n \text{ is an integer.}$$

Example 4 – *Solution*

cont'd

See Figure P.44.



Solution points of $\sin\theta = -\frac{\sqrt{3}}{2}$

Figure P.44

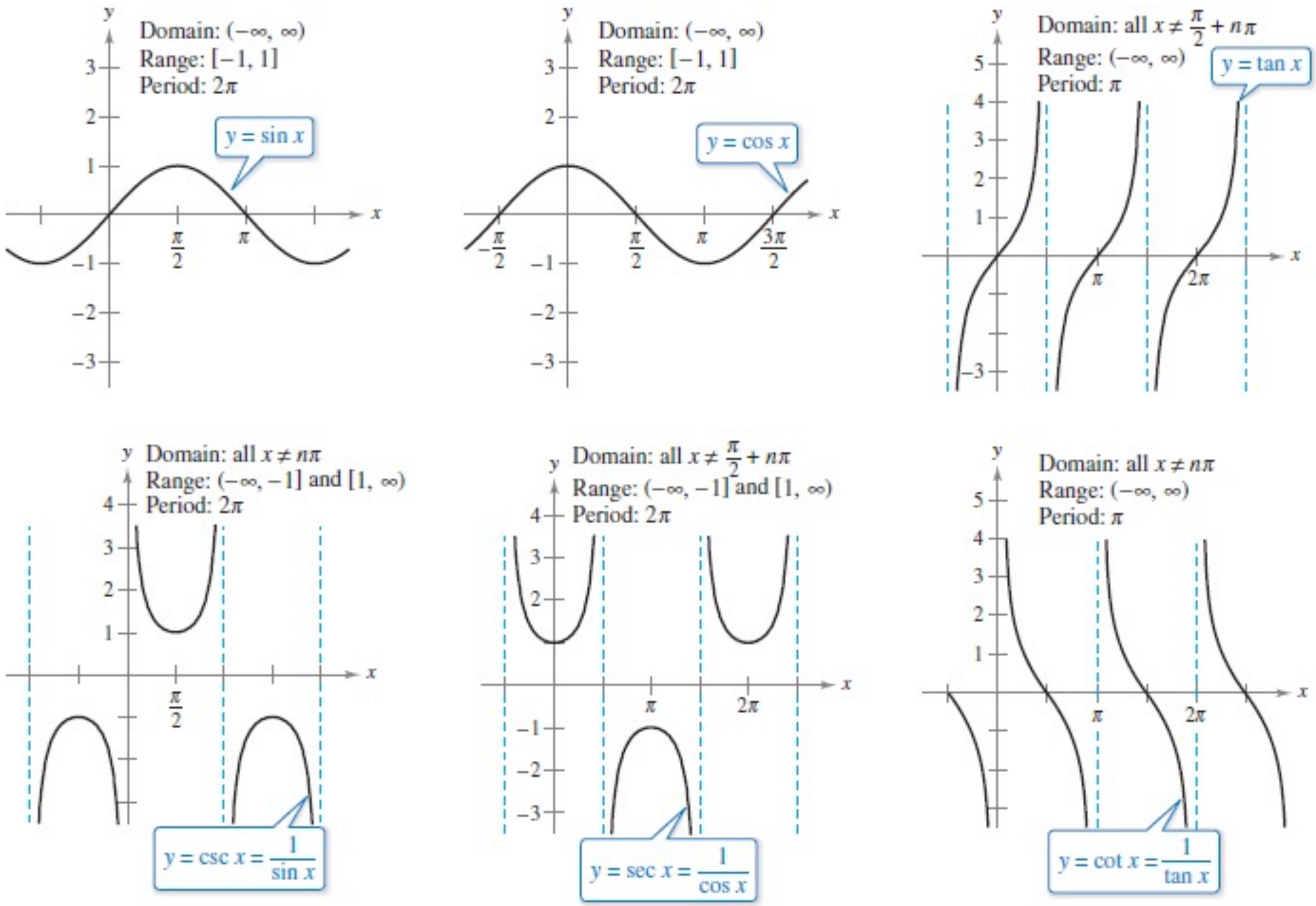
Graphs of Trigonometric Functions

Graphs of Trigonometric Functions

A function f is **periodic** when there exists a positive real number p such that $f(x + p) = f(x)$ for all x in the domain of f.

The least such positive value of p is the **period** of f. The sine, cosine, secant, and cosecant functions each have a period of 2π, and the other two trigonometric functions, tangent and cotangent, have a period of π, as shown in Figure P.45.

Graphs of Trigonometric Functions



The graphs of the six trigonometric functions

Figure P.45

Graphs of Trigonometric Functions

Note in Figure P.45 that the maximum value of sin x and cos x is 1 and the minimum value is −1.

The graphs of the functions $y = a \sin bx$ and $y = a \cos bx$ oscillate between $-a$ and a, and so have an **amplitude** of $|a|$.

Furthermore, because $bx = 0$ when $x = 0$ and $bx = 2\pi$ when $x = 2\pi/b$, it follows that the functions $y = a \sin bx$ and $y = a \cos bx$ each have a period of $2\pi/|b|$.

Graphs of Trigonometric Functions

The table below summarizes the amplitudes and periods of some types of trigonometric functions.

Function	Period	Amplitude
$y = a \sin bx$ or $y = a \cos bx$	$\frac{2\pi}{\lvert b\rvert}$	$\lvert a\rvert$
$y = a \tan bx$ or $y = a \cot bx$	$\frac{\pi}{\lvert b\rvert}$	Not applicable
$y = a \sec bx$ or $y = a \csc bx$	$\frac{2\pi}{\lvert b\rvert}$	Not applicable

Example 6 – *Sketching the Graph of a Trigonometric Function*

Sketch the graph of $f(x) = 3 \cos 2x$.

Solution:

The graph of $f(x) = 3 \cos 2x$ has an amplitude of 3 and a period of $2\pi/2 = \pi$.

Using the basic shape of the graph of the cosine function, sketch one period of the function on the interval $[0, \pi]$, using the following pattern.

Maximum: $(0, 3)$

Minimum: $\left(\frac{\pi}{2}, -3\right)$

Maximum: $(\pi, 3)$

Example 6 – *Solution*

cont'd

By continuing this pattern, you can sketch several cycles of the graph, as shown in Figure P.46.

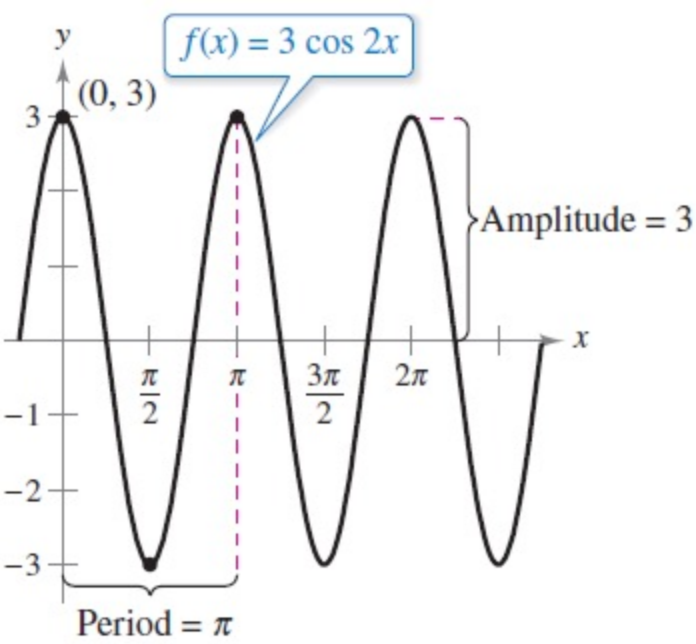
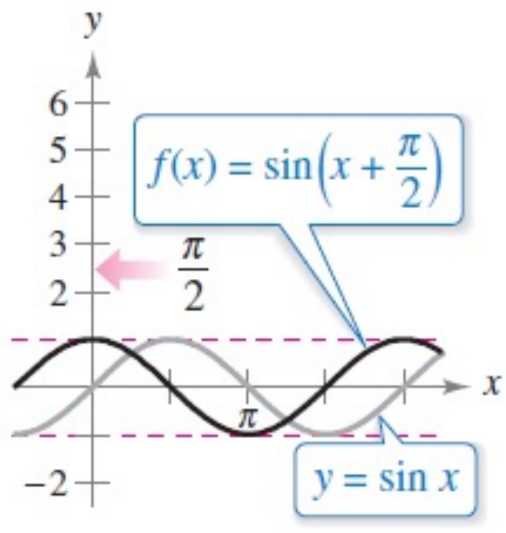


Figure P.46

Example 7 – *Shifts of Graphs of Trigonometric Functions*

a. To sketch the graph of $f(x) = \sin(x + \pi/2)$, shift the graph of $y = \sin x$ to the left $\pi/2$ units, as shown in Figure P.47(a).



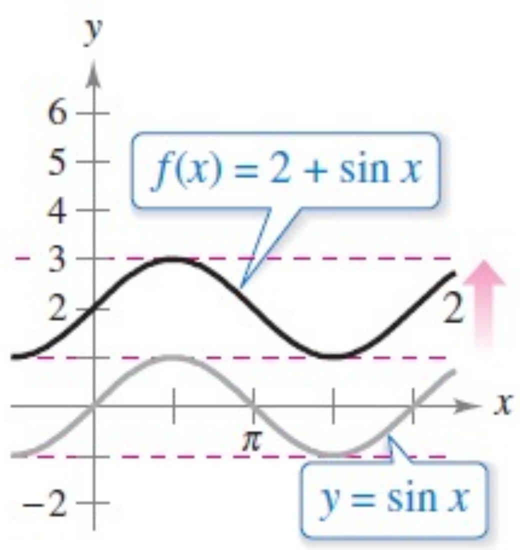
(a) Horizontal shift to the left

Transformations of the graph of $y = \sin x$

Figure P.47

Example 7 – *Shifts of Graphs of Trigonometric Functions* cont'd

b. To sketch the graph of $f(x) = 2 + \sin x$, shift the graph of $y = \sin x$ upward two units, as shown in Figure P.47(b).



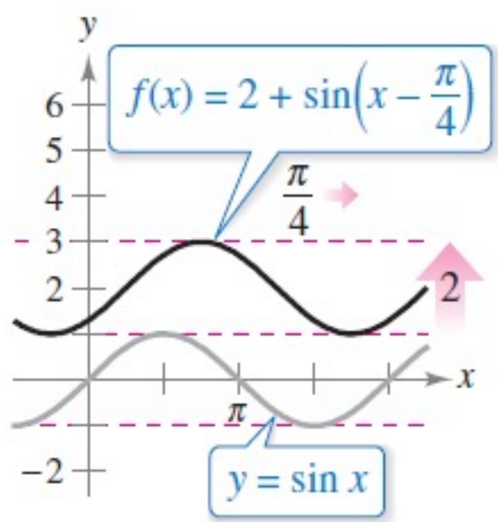
(b) Vertical shift upward

Transformations of the graph of $y = \sin x$

Figure P.47

Example 7 – *Shifts of Graphs of Trigonometric Functions* cont'd

c. To sketch the graph of $f(x) = 2 + \sin(x - \pi/4)$, shift the graph of $y = \sin x$ upward two units and to the right $\pi/4$ units, as shown in Figure P.47(c).



(c) Horizontal and vertical shifts

Transformations of the graph of *y* = sin *x*

Figure P.47